Math 105 Chapter 8: Separable Differential Equations

One of the main applications of calculus is to use derivatives of a function to determine more information about our function.

Most times in real life you aren't given explicity what the function or the derivative is, but rather how they relate to each other. That relation is a differential equation. More formally,

Definition: A differential equation is a mathematical equation for an unknown function f that relates the values of the function itself to it's derivatives of various orders.

eg $y' = 5$, $y' + 5y = \cos x$, $(y')^2 = x + y$, $y'' = -ky$

The goal when studying DE's is to determine what y is.

They are worth studying because they simply put they govern the rules of how the world around us work. In general they can be very difficult to solve or even impossible. If we wanted to, we could spend our entire lives studying them and barely scratch the surface (and some peope actually do!).

Definition: Given 2 quantities A, B, we say A is proportional to B if there is a $K \in \mathbb{R}$ such that

$$A = KB$$

we write this as

$$A \propto B$$

alpha

eg When modeling the growth of money in your saving account we have continuously compounded, one has the rate at which your money grows is proportional to the amount of money you have in your account.

Find a function describing how your savings evolve over time.

Sol^n Let $P(t)$ denote your savings in your account at time t.

We are given, rate of change of $P(t)$ is proportional to P, so

$$P'(t) \propto P$$

$$\Rightarrow \quad P'(t) = \delta P \quad \text{for some } \delta \in \mathbb{R}.$$

So our differential equation is $P'(t) = \delta P$. Lets solve.

$$\frac{P'(t)}{P} = \delta$$

Now lets integrate

$$\int \frac{P'(t)}{P(t)}\, dt = \int \delta \, dt$$

$$\int \frac{P'(t)\, dt}{P(t)} = \int \frac{dP}{P} = \log|P| + C_1$$

$$\int \delta \, dt \, dt = \delta t + C_2$$

$$\Rightarrow \quad \log|P| = \delta t + C, \quad C = C_2 - C_1$$

So $P(t) = \pm e^{\delta t + c}$

$= Ae^{\delta t}$, $A = \pm e^{c}$

So we have $Ae^{\delta t}$ is a solution to the DE for all A.

In real life we only have one solution how do we rectify this?

Well if I tell you how much money I have at time 0, say

$$P(0) = P_0$$

Then $A = P_0$ and thus

$$P(t) = P_0 e^{\delta t}$$

Note: In finance δ is called the "force of interest" and the annual interest rate, i, satisfies

$$1 + i = e^{\delta}$$

The above motivates the following definition:

Definition An initial value proplem (IVP) is a DE, together with a specified initial value.

eg $\frac{dP}{dt} = \delta P$, $P(0) = P_0$

Definition: The order of a differential equation is highest order derivative that appears in the DE.

eg $y' = \delta y$ is a first order differential equation.

$y'' + y' = yx$ is a second order differential equation.

The general solution of a first order differential equation will have one arbitrary constant that arises from the constant of integration.

eg $\frac{dy}{dx} = \delta y \Rightarrow y(x) = A e^{\delta x}$, for all A

When you specify the initial condition we find a unique value for the constant.

eg $y(0) = 5 \Rightarrow A = 5$ & $y(x) = 5e^{\delta x}$

Let us reexamine the previous example. Suppose as before, $P(t)$ represents the amount of money into your savings account. The rate of change of your savings is proportional to how much you have in your account and suppose you add an addition k dollars per year.

So $\frac{dP}{dt}$ increases by k, implying

$$\frac{dP}{dt} = \delta P + k, \quad P(0) = P_0$$

Let us solve this IVP.

Sol^n

$$\frac{dP}{dt} = \delta P + K$$

$$\Rightarrow \frac{dP}{\delta P + K} = dt$$

$$\Rightarrow \int \frac{dP}{\delta P + K} = \int dt$$

Remark: The first step should seem dubious, since it is. We can't can multiply/divide "dP", "dt" in general but in this case it is justified by the following correct solution.

$$\frac{dP}{dt} = \delta P + K$$

$$\Rightarrow \frac{P'(t)}{\delta P(t) + K} = 1$$

$$\Rightarrow \int \frac{P'(t)\,dt}{\delta P(t) + K} = \int dt \quad \text{, by integrating both sides wrt } t.$$

$$\Rightarrow \int \frac{dP}{\delta P + K} = \int dt \quad \text{, since } P = P(t),\ dP = P'(t)dt$$

So we will save some time and just "multiply" the dt, without remorse.

Now let us continue.

$$\int \frac{dP}{\delta P + k} = \int dt$$

$$\Rightarrow \quad \frac{1}{\delta} \log|\delta P + k| = t + C_1$$

$$\Rightarrow \quad \log|\delta P + k| = \delta t + C_2, \qquad C_2 = \delta C_1$$

$$\Rightarrow \quad \delta P + k = \pm e^{\delta t + C_2}$$
$$= \pm e^{C_2} e^{\delta t}$$

$$\Rightarrow \quad P(t) = \frac{\pm e^{C_2} e^{\delta t} - k}{\delta}$$

$$= A e^{\delta t} - \frac{k}{\delta}, \qquad A = \frac{\pm e^{C_2}}{\delta}$$

Now we want to find what A is, so we we use the

$$P(0) = P_0 \quad \Rightarrow \quad P_0 = A - \frac{k}{\delta}$$

$$\Rightarrow \quad A = P_0 + \frac{k}{\delta}$$

$$\text{So } P(t) = \left(P_0 + \frac{k}{\delta}\right) e^{\delta t} - \frac{k}{\delta}$$

$$= \underbrace{P_0 e^{\delta t}}_{(1)} + \underbrace{\frac{k}{\delta}\left(e^{\delta t} - 1\right)}_{(2)}$$

(1) represents the growth of your account without any additional money being added.

(2) Represents the additional growth/decay in your account from the additional money being added. So even though you add money into your acount at a constant rate the amount grows exponentially over time.

Similarly if $k < 0$ (ie you remove money from your acount) then the saving decrease at an exponential rate.

- Let us see what happens to the savings as $\delta \to 0$.

$$\lim_{\delta \to 0} P(t) = \lim_{\delta \to 0} P_0 e^{\delta t} + k \lim_{\delta \to 0} \frac{e^{\delta t} - 1}{\delta}$$

$$= P_0 + k \lim_{\delta \to 0} \frac{te^{\delta t} - 0}{1} \quad , \text{ by l'hopital}$$

$$= P_0 + kt$$

Which is the solution to the original DE by taking $\delta \to 0$.

$$\frac{dP}{dt} = k \quad , \quad P(0) = P_0$$

Definition: We have a separable differential equation is of the form

$$\frac{dy}{dx} = f(x)\,g(y)$$

for some $f(x)$, $g(x)$.

eg $\dfrac{dy}{dx} = \delta y + k$ is seperable since

$$\frac{dy}{dx} = 1\,(\delta y + k) \quad , \text{ so } \quad f(x) = 1 \quad , \quad g(y) = \delta y + k$$

eg $\dfrac{dy}{dx} = x e^{y} + x$ is separable since

$$\frac{dy}{dx} = x\,(e^{y} + 1) \quad , \text{ so } \quad f(x) = 1 \quad , \quad g(y) = e^{y} + 1$$

eg $\dfrac{dy}{dx} = y + x$ is <u>not</u> separable

Lets modify the previous example. One assumption we are making is that the interest rate is constant. But in reality it is constantly changing.

So suppose $\delta = \delta(t)$ ie, δ is a function of time, and we do not add or take away any money. So

$$\frac{dP}{dt} = \delta(t)\,P \quad , \quad P(0) = P_0$$

Sol^n This a seperable DE with $f(t) = \delta(t)$, $g(P) = P$

So $\frac{dP}{dt} = \delta(t) P$

$\Rightarrow \frac{dP}{P} = \delta(t) dt$

$\int \frac{dP}{P} = \int \delta(t) dt$

$\log|P| = \int_0^t \delta(s) ds + C$

$|P| = e^{\int_0^t \delta(s) ds + c}$

$P(t) = \pm e^c e^{\int_0^t \delta(s) ds}$

$P(t) = A e^{\int_0^t \delta(s) ds}$, $A = \pm e^c$

$P(0) = P_0 \Rightarrow A = P_0$

So $P(t) = P_0 e^{\int_0^t \delta(t) dt}$

So if $\delta(t)$ is increasing linearly for example

$\delta(t) = 1 + 0.02t$

ie initially it is 1% and increasies 0.02% every year

$\Rightarrow P(t) = P_0 e^{\int_0^t 1 + 0.02 s\, ds}$

$= P_0 e^{\int_0^t 1 + 0.02 s\, ds}$

$= P_0 e^{[s + 0.01 s^2]_0^t}$

$= P_0 e^{t + 0.01 t^2}$

Let's do another example.

eg $\quad y'(x) = 1 + y^2 - \sin^2 x - \sin^2 x \, y^2 \quad ; y(0) = 1$

Sol^n $\quad \frac{dy}{dx} = 1 + y^2 - \sin^2 x (1 + y^2)$

$$= (1 - \sin^2 x)(1 + y^2)$$

$$= \cos^2 x (1 + y^2)$$

Thus the DE is seperable

$$\Rightarrow \quad \frac{dy}{1+y^2} = \cos^2 x \, dx$$

$$\Rightarrow \quad \underbrace{\int \frac{dy}{1+y^2}}_{\parallel} = \underbrace{\int \cos^2 x \, dx}_{\parallel}$$

$$\arctan y = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

So $\quad y(x) = \tan\left(\frac{x}{2} + \frac{\sin(2x)}{4} + C\right)$

So to solve for C, we use the initial condition.

$$y(0) = 1 \Rightarrow 1 = \tan\left(\frac{0}{2} + \frac{\sin(2 \cdot 0)}{4} + C\right)$$

$$= \tan(C)$$

So $C = \frac{\pi}{4}$ and $y(x) = \tan\left(\frac{x}{2} + \frac{\sin(2x)}{4} + \frac{\pi}{4}\right)$

eg Suppose you want to spread a rumor that Saif has cooties, in your class. Let N denote the number of number of people that know the rumor. The more people that know, the faster the rumor will spread.

$$\text{So} \quad N'(t) = k N(t) \quad \text{for some } A.$$

But we have that the more students that know, the less people we have to spread the rumor to. That If there are N_0 amount of people in the class

$$N'(t) = k N (N_0 - N) \quad , N(0) = 1 \leftarrow \text{since only you know I have cooties at time 0;}$$

This is called the logistic equation.

Sol^n

$$\frac{dN}{dt} = k N (N_0 - N)$$

$$\int \frac{dN}{N(N_0 - N)} = \int k\, dt$$

exercise: show: $N(t) = N_0 \left(\dfrac{1}{1 + Ce^{-kN_0 t}} \right)$, $C \in \mathbb{R}$

Letting $N(0) = 1 \Rightarrow 1 = N_0 \left(\dfrac{1}{1 + C} \right)$

$$\Rightarrow C = N_0 - 1$$

So $N(t) = N_0 \left(\dfrac{1}{1 + (N_0 - 1) e^{-kN_0 t}} \right)$